Ergodic Properties for Multirate Linear Systems

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Ergodicity

1. A discrete-time random process $x = \{x(t)\}_{t \in \mathbb{Z}}$ is said to be **ergodic in the mean** if

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} x(t) - \mathcal{E}\{x(t)\} \overset{w.p.1}{=} 0
$$

2. Two random processes $x$ and $y$ are said to be **jointly ergodic in the correlation** if, for every $\tau \in \mathbb{Z}$,

$$
\lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} x^*(t)y(t+\tau) - \mathcal{E}\{x^*(t)y(t+\tau)\} \overset{w.p.1}{=} 0
$$

3. A random process is **ergodic in the correlation** if it is jointly ergodic in the correlation with itself.
Multirate and Linear Systems

A multirate linear system is a combination of the following basic operations:

1. The **addition** of two signals.

2. Filtering with a **uniformly stable linear filter**, i.e., a time-varying linear filter with impulse response \( \{h_t(\tau) : t, \tau \in \mathbb{Z}\} \) for which there exists \( h \in l_1(\mathbb{Z}) \) such that, for all \( \tau \in \mathbb{Z} \),

\[
|h_t(\tau)| \leq h(\tau) \quad \text{for all} \quad t \in \mathbb{Z}
\]

3. The operation of **downsampling** by a factor \( D \)

\[
y(t) = x(Dt)
\]

4. The operation of **upsampling** by a factor \( D \)

\[
y(t) = \begin{cases} 
x(t/D); & t/D \in \mathbb{Z} \\
0; & t/D \notin \mathbb{Z}
\end{cases}
\]
Research Problem

It can be shown that in a multirate linear system, the ergodicity in the mean can be lost after:

1. Downsampling.

2. Uniformly stable linear filtering.

and the ergodicity in the correlation can be lost after:

1. Downsampling. (Example 1)

2. Uniformly stable linear filtering. (Example 2)

3. Addition of two signals that are ergodic in the correlation. (Example 3)

The above implies that the ergodicity of all the signals in a multirate linear system cannot be guaranteed by simply assuming that its input signals are ergodic. For the stochastic analysis of a multirate linear system, the ergodicity of its signals is a desirable property. To cope with this issue we introduce the definitions of strong ergodicity in the mean and strong ergodicity in the correlation.
Strong Ergodicity in the Mean

The random process $x$ is called strongly ergodic in the mean if the following conditions hold

(M1) $x$ is ergodic in the mean.

(M2) If the random process $y$ is also strongly ergodic in the mean, then $x + y$ is strongly ergodic in the mean.

(M3) The filtering of $x$ by a uniformly stable linear filter, yields a random process that is strongly ergodic in the mean.

(M4) The downsampling of $x$ by any factor yields a random process that is strongly ergodic in the mean.

(M5) The upsampling of $x$ by any factor yields a random process that is strongly ergodic in the mean.
About the Definition of Strong Ergodicity in the Correlation

Condition (M2) says: If \( x \) and \( y \) are strongly ergodic in the mean, then

\[ x + y \] is strongly ergodic in the mean

But we can find (Example 4) \( u \) and \( v \) independent, such that,

\[ u + v \] is not ergodic in the mean

So, Including a condition similar to (M2) in the definition of strong ergodicity in the correlation would exclude the independent signals \( \Rightarrow \) would not be of much interest.

In order to relax the definition we require: if \( x \) and \( y \) are strongly ergodic in the correlation and jointly strongly strongly ergodic in the correlation, then

\[ x + y \] is strongly ergodic in the mean

So strong ergodicity in the correlation becomes a joint condition of two signals.
Strong Ergodicity in the Correlation

Two random processes $x$ and $y$ are said to be jointly strongly ergodic in the correlation if:

(C0) $y$ and $x$ are jointly strongly ergodic.
(C1) $x$ and $y$ are jointly ergodic in the correlation.
(C2) If the random process $z$ is jointly strongly ergodic in the correlation with both $x$ and $y$, then $z$ is also jointly strongly ergodic in the correlation with $x + y$.
(C3) The filtering of $y$ by a uniformly stable linear system, yields a random process that is jointly strongly ergodic in the correlation with $x$.
(C4) The downsampling of $y$ by any factor yields a random process that is jointly strongly ergodic in the correlation with $x$.
(C5) The upsampling of $y$ by any factor yields a random process that is jointly strongly ergodic in the correlation with $x$.

A random process is called strongly ergodic in the correlation if it is jointly strongly ergodic in the correlation with itself.
Main Result

**Theorem:** Let $v$ and $w$ be random processes formed from the random processes of the finite collection $u_m$, $m = 1, \cdots, M$, by any finite combination of additions, uniformly stable linear filtering, downsampling and upsampling. Then:

1. If, for every $m$, the random process $u_m$ is uncorrelated and has uniformly bounded second moments, then $v$ is **ergodic in the mean**.

2. If the random processes $\{u_m : m = 1, \cdots, M\}$ are mutually independent and have uniformly bounded fourth moments, then $v$ and $w$ are **jointly ergodic in the correlation**.

**Remarks:**

1. The signals $u_m$ can include deterministic signals in $l_\infty(\mathbb{Z})$.

2. If the random processes $u_m$ have Gaussian distribution, then the uniform boundedness of the fourth moments is equivalent to that of the second moments.
Main Result

Uncorrelated, uniformly bounded second moments

\[ u_1(t), u_2(t), \ldots, u_m(t) \]

Multirate Linear System

Ergodic in the mean

\[ v(t) \]

Mutually independent, uniformly bounded fourth moments

\[ u_1(t), u_2(t), \ldots, u_m(t) \]

Multirate Linear System

Jointly ergodic in the correlation

\[ v(t), w(t) \]
Sketch of the Proof

**Definition:** Let $\xi$ be a random process and $T \in \mathbb{N}$. Define

$$\|\xi\|_S = \sup_{T \in \mathbb{N}, d \in \mathbb{Z}} \left( \frac{1}{T} \sum_{t,s=1}^{T} |E\{\xi(s+d)\bar{\xi}(t+d)\}|^2 \right)^{1/4}$$

and

$$S(\Omega, \mathcal{A}, \mathbb{P}) = \{\xi : \|\xi\|_S < \infty\}$$

**Lemma:** The operator $\|\cdot\|_S$ defines a norm on $S(\Omega, \mathcal{A}, \mathbb{P})$, i.e., for any $\xi, \zeta \in S(\Omega, \mathcal{A}, \mathbb{P})$ and $c \in \mathbb{C}$, the following conditions hold:

\begin{enumerate}
  \item[(N1)] $\|c\xi\|_S = |c| \|\xi\|_S$;
  \item[(N2)] $\|\xi + \zeta\|_S \leq \|\xi\|_S + \|\zeta\|_S$;
  \item[(N3)] $\|\xi\|_S = 0 \Rightarrow \xi = 0$.
\end{enumerate}

**Lemma:** Let $\xi$ be random process and let $\zeta$ be a random process generated from $\xi$ by downsampling, upsampling or filtering via a uniformly stable linear filter. If $\xi \in S(\Omega, \mathcal{A}, \mathbb{P})$, then $\zeta \in S(\Omega, \mathcal{A}, \mathbb{P})$. 

Sketch of the Proof

Sufficient condition for strong ergodicity in the mean: Let $x$ be a random process. Define the random processes $\xi_x$ by

$$\xi_x(t) = x(t) - \mathcal{E}\{x(t)\}$$

If

$$\xi_x \in \mathcal{S}(\Omega, \mathcal{A}, \mathbb{P})$$

then $x$ is strongly ergodic in the mean.

Corollary: If a random process is uncorrelated and has uniformly bounded second moments, then it is strongly ergodic in the mean.
Sketch of the Proof

**Sufficient condition for strong ergodicity in the correlation:** Let \( x \) and \( y \) be random processes. Define the collection of random processes \( \xi_{x,y}(A, B, a, b), (A, B, a, b) \in \mathbb{N}^2 \times \mathbb{Z}^2 \) by

\[
\xi_{x,y}(A, B, a, b)(t) = x^*(At + a)y(Bt + b) - \mathcal{E}\{x^*(At + a)y(Bt + b)\}
\]

If there exists \( C > 0 \) such that

\[
\|\xi_{x,y}(A, B, a, b)\|_S < C \text{ for all } (A, B, a, b)
\]

then \( x \) and \( y \) are jointly strongly ergodic in the correlation.

**Corollary:** If two random processes are mutually independent and have uniformly bounded fourth moments, then they are jointly strongly ergodic in the correlation.
The signals $u$ and $y$ are split into $M$ subbands. The parametric subband model

$$\hat{G}(q, \theta) = \text{diag}\{\hat{G}_m(q, \theta_m), \ m = 1, \cdots, M\}$$

is an FIR diagonal transfer matrix. The set of parameters on the $m$-th channel after sample $N$ is

$$\theta_{m,N} \in \text{arg min}_{\theta_m} \frac{1}{N} \sum_{t=1}^{N} |\hat{V}_m(t, \theta_m)|^2$$

The error signal is reconstructed by passing $\hat{V}(t, \theta_m)$ through a synthesis filterbank.
For a given set of parameters $\theta_m$, the power of the signal $\hat{V}_m(t, \theta_m)$ is defined by

$$S_{\hat{V}_m}(\theta_m) := \lim_{T \to \infty} \frac{1}{T} \sum_{t=1}^{T} \mathcal{E}\{|\hat{V}_m(t, \theta_m)|^2\}$$

We say that there is strong and optimum convergence in the subband $m$ if

$$\lim_{N \to \infty} S_{\hat{V}_m}(\theta_m, N) \overset{w.p.1}{=} \min_{\theta_m \in \mathcal{D}_m} S_{\hat{V}_m}(\theta_m)$$

In order to have strong and optimum convergence in all the subband we need that, for all $m = 1, \cdots, M$,

$U_m$ and $Y_m$ are ergodic and jointly ergodic in the correlation.

In view of the Theorem, this condition is guaranteed if the input signals $u$ and $v$ are assumed to be generated from mutually independent random processes and bounded deterministic signals, by any combination of uniformly-stable linear filtering, downsampling, upsampling and additions.
Possible Further Work

To prove that a random process is ergodic in the mean is essentially to verify the Strong Law of Large Numbers (SLLN). There are two approaches to verify the SLLN:

1. **Kolmogorov’s SLLN**: Requires that the samples are independent and strictly stationary.
   - A generalization that relax the independence condition is Birkhoff’s ergodic theorem, which is the cornerstone of the ergodic theory:
     - It provides conditions for ergodicity in a very general sense (including the mean and correlation).
     - These conditions are preserved if the random process is transformed via any measurable (nonlinear) stationary transformation ⇒ is not suitable for our purposes because it excludes time-variant operations such as upsampling, downsampling and time-variant filtering.
• In a further generalization of this approach [Gray, 1988] the requirement for strict stationarity is relaxed to the so-called asymptotic mean stationarity (AMS). This generalization is not sufficient for our purposes because the AMS condition can be destroyed by time-variant operations (Example 6).

2. **Rajchman’s SLLN:** Requires that the samples are uncorrelated and have uniformly bounded second moments, i.e., there are no stationarity conditions

• Our work consisted on generalizing the Rachman’s SLLN to relax the independence of the samples, to consider ergodicity in the correlation and to make it invariant under multirate linear transformations.

It remains a challenging problem to find a unified approach to general ergodicity analysis for random processes in multirate signal processing. I.e., to find conditions for ergodicity in a general sense, which are invariant under non-linear, time-variant transformations.